Kinetic description of quasi-stationary axisymmetric collisionless accretion disk plasmas with arbitrary magnetic field configurations

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A kinetic treatment is developed for collisionless magnetized plasmas occurring in high-temperature, low-density astrophysical accretion disks, such as are thought to be present in some radiatively-inefficient accretion flows onto black holes. Quasi-stationary configurations are investigated, within the framework of a Vlasov-Maxwell description. The plasma is taken to be axisymmetric and subject to the action of slowly time-varying gravitational and electromagnetic fields. The magnetic field is assumed to be characterized by a family of locally nested but open magnetic surfaces. The slow collisionless dynamics of these plasmas is investigated, yielding a reduced gyrokinetic Vlasov equation for the kinetic distribution function. For doing this, an asymptotic quasi-stationary solution is first determined, represented by a generalized bi-Maxwellian distribution expressed in terms of the relevant adiabatic invariants. The existence of the solution is shown to depend on having suitable kinetic constraints and conditions leading to particle trapping phenomena. With this solution one can treat temperature anisotropy, toroidal and poloidal flow velocities and finite Larmor-radius effects. An asymptotic expansion for the distribution function permits analytic evaluation of all of the relevant fluid fields. Basic theoretical features of the solution and their astrophysical implications are discussed. As an application, the possibility of describing the dynamics of slowly time-varying accretion flows and the self-generation of magnetic field by means of a "kinetic dynamo effect" is discussed. Both effects are shown to be related to intrinsically-kinetic physical mechanisms.

I. INTRODUCTION

This paper is part of an investigation concerning the theoretical formulation of kinetic theory for collisionless astrophysical plasmas in accretion disks (ADs) around compact objects, and its application to the study of their equilibrium properties and dynamical evolution. Note that what is meant here by the word "equilibrium" is in general a stationary-flow solution, which can also include a stationary radial accretion velocity.

In contrast with the majority of previous treatments, which are based on fluid approaches within the context of hydrodynamics (HD) or magnetohydrodynamics (MHD) [1–5], here we adopt a kinetic approach. This provides a phase-space treatment allowing us to formulate a consistent description of plasma dynamics. Kinetic theory is essential for studying both stationary configurations and dynamical evolution of plasmas when kinetic effects are relevant, such as ones associated with conservation of particle adiabatic invariants, temperature anisotropy, finite Larmor-radius (FLR) effects (as pointed out in Ref.[6]) and kinetic trapping phenomena. These properties are relevant for magnetized plasmas and in particular for those arising in ADs [7–9] whenever the plasma is regarded as collisionless or weakly collisional [6, 10].

In the context of astrophysical ADs, there are several examples of collisionless plasmas of this kind, with both strong and weak magnetic fields. One is the case of radiatively inefficient accretion flows (RIAFs) [11, 12], in geometrically thick disks around black holes consisting of two-temperature plasma, with the ion temperature being much higher than the electron one, and the timescale of the Coulomb collision frequency being much longer than the inflow time. Other interesting applications occur in ADs around neutron stars and white dwarfs: in the inner regions of such disks,

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where the magnetic field of the central object becomes dominant, ions and electrons can be collisionally decoupled and sustain different temperatures. This happens, in particular, if the radiative cooling time-scale of the electrons is much shorter than the time-scale for electron-ion collisions. In this way, electrons and ions are thermally decoupled: the two charged species acquire unequal temperatures and the accretion flow becomes a two-temperature flow [13, 14].

In our earlier paper (Ref.[6], hereafter referred to as Paper I), we presented preliminary results in this direction, concerning formulation of kinetic theory for investigating stationary solutions for collisionless AD plasmas, focusing on configurations with locally-closed magnetic flux surfaces. The present paper is intended as a continuation of the previous one, with the aim of generalizing the previous solution to arbitrary magnetic field configurations, which are no longer restricted to localized spatial domains in the disk. We refer to Fig.1 below and the discussion in Section II for an explicit comparison of the two configurations. More specifically, in Paper I the treatment concerned collisionless magnetized plasmas characterized by locally closed and nested magnetic surfaces. For such configurations, it was shown that suitable kinetic distribution functions (KDFs) are permitted, describing both kinetic and gyrokinetic (GK) equilibria (see definition in Paper I), which are represented by generalized Maxwellian and bi-Maxwellian KDFs. A main feature was the inclusion in the kinetic treatment of both temperature anisotropy and FLR effects. In particular, in Paper I and in Ref.[15], it was proved that these equilibria can sustain a stationary kinetic dynamo. As a basic consequence, it was found that both toroidal and poloidal equilibrium magnetic fields can be self generated for quasi-neutral plasmas, *without ongoing instabilities and/or turbulence phenomena*. In particular, in closed nested field configurations [and hence in the local absence of net accretion flow], the toroidal field component was found to be produced by diamagnetic effects driven by the species temperature anisotropies. As a further development, in Paper I and in Ref.[16], it was pointed out that the kinetic treatment allows one to construct exact fluid equilibria (identically satisfying the corresponding fluid equations). Using a perturbative expansion, a well-defined set of kinetic closure conditions was determined analytically for the relevant stationary moment equations.

A. Accretion disks in astrophysics

Despite more than forty years of observations and theoretical investigations, there is a lot remaining to be understood about the physical processes governing the structure and evolution of ADs. They are observed in a wide range of astrophysical contexts [1] and consist of plasma orbiting a central object with the velocities of the inward accretion flow usually being much smaller than the rotational velocities. In order for the accretion to happen, there needs to be a net outward transport of angular momentum and there are several conceivable mechanisms for producing this (see for example [17, 18]). The most obvious one is fluid viscosity, but this would need to be an "anomalous" viscosity, driven by some type of turbulence, rather than a standard viscosity connected with Coulomb collisions (Spitzer viscosity) which would be much too small to explain the observed accretion rates under the conditions actually found in accretion disks (see [19] for a review of turbulence mechanisms). However, other collisionless physical mechanisms are possible in principle, such as kinetic instabilities, radiation effects and magnetic reconnection. The aim of the present paper is to help in preparing the way for a discussion of these. We focus on AD plasmas immersed in slowly time-varying magnetic fields, characterized locally by open nested magnetic surfaces. For these systems, no kinetic treatment has been available up to now. The origin of their magnetic fields varies depending on the type of the central object: in the case of black holes, the fields are only ones self-generated by currents in the plasma itself via dynamo effects, while with neutron stars and white dwarfs there can also be a magnetic field intrinsic to the central object [1, 2]. The interplay between magnetic fields and accretion plasmas can affect the overall velocity profile of the disk, as well as giving rise to species-dependent velocities and rotational frequencies [4, 5, 10]. Moreover, the magnetic field can be a source of anisotropies in the KDF and allow particular symmetries which influence both the single particle and collective plasma behavior. The transport of angular momentum, the accretion flow and the possible generation of jets [20–22] are all strongly dependent on the magnetic field structure and so magnetic fields play an important role for AD physics.

B. Goals and scheme of the presentation

The purpose of this paper is to formulate a comprehensive kinetic treatment for collisionless axisymmetric AD plasmas including both accretion flows and collisionless dynamo effects. We include general relative orderings between the magnitudes of the external and self-generated magnetic fields and allow the magnetic field be non-uniform and slowly time-varying while possessing locally nested open magnetic surfaces.

Extending the investigation developed in Paper I, we do this by constructing particular quasi-stationary solutions of the Vlasov-Maxwell equations, characterized by generalized bi-Maxwellian phase-space distributions (see also Paper I), which are referred to here as quasi-stationary asymptotic KDFs (QSA-KDFs). As discussed below, the functional form of these solutions is physically motivated. We will show that this makes possible the explicit inclusion of both *temperature anisotropies* and *parallel velocity perturbations* in the QSA-KDFs (see the definition below in Section 4). This is done, first, by developing an "ad hoc" formulation for GK theory in the presence of a gravitational field, making it possible to directly construct the relevant particle guiding-center adiabatic invariants. The QSA-KDFs are then expressed in terms of these. Remarkably, this allows also the consistent treatment of trapping phenomena due to spatial variations both of the magnetic field and of the total effective potential (gravitational EM trapping). Second, the QSA-KDFs are constructed by imposing appropriate *kinetic constraints* (see Section 4), requiring that suitable *structure functions* (see below) which enter the definition of the QSA-KDFs, depend only on the azimuthal canonical momentum and total particle energy. By invoking suitable perturbative expansions, it follows that the relevant moments and moment equations can be evaluated analytically. The solution thus obtained can be used for investigating the quasi-stationary dynamics of magnetized AD plasmas, including description of quasi-stationary accretion flows and "kinetic dynamo effects" allowing for the generation of finite poloidal and toroidal magnetic fields. In particular, the kinetic theory predicts the possibility of pure matter inflows as well as the independent coexistence of both inflows and outflows.

The paper is organized as follows. In Section 2 we summarize the basic assumptions and definitions of the theory. In Section 3 we formulate the GK theory for magnetized accretion disk plasmas, deriving the relevant integrals of motion and guiding-center adiabatic invariants and discussing the particle trapping phenomenon. Section 4 deals with the construction of a generalized asymptotic stationary KDF, with the inclusion of parallel velocity perturbations and the adoption of suitable kinetic constraints. In Section 5 we give an analytic expansion for the KDF and discuss its main features. Section 6 deals with the relationship between kinetic theory and the corresponding fluid treatment, which concerns the validity of moment equations and the analytic calculation of fluid fields. Section 7 is dedicated to discussing the temporal evolution of the GK equilibria and derivation of the dynamical equation for the GK KDF. In Section 8 we investigate the implication of the kinetic solution for the Ampere equation and the existence of the kinetic dynamo effect. Then, in Section 9 we discuss the treatment of quasi-stationary accretion flow within the present formulation, showing that solutions with net radial accretion are admitted consistently with the constraints imposed by the Maxwell equations. Finally, Section 10 contains a summary of the main results with closing remarks.

II.   BASIC ASSUMPTIONS AND DEFINITIONS

Ignoring possible weakly-dissipative effects (Coulomb collisions and turbulence), we shall assume that the KDF and the EM fields associated with the plasma obey the system of Vlasov-Maxwell equations, with Maxwell's equations being considered in the quasi-static approximation. For definiteness, we shall consider here a plasma consisting of at least two species of charged particles: one species of ions (i) and one of electrons (e).

Following the treatment presented in Paper I, we shall take the AD plasma to be: a) *non-relativistic*, in the sense that it has non–relativistic species flow velocities, that the gravitational field can be treated within the classical Newtonian theory, and that the non-relativistic Vlasov kinetic equation is used as the dynamical equation for the KDF; b) *collisionless*, so that the mean free path of the plasma particles is much longer than the largest characteristic scale length of the plasma; c) *axisymmetric*, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the azimuthal angle φ, when referred to a set of cylindrical coordinates (R, φ, z); d) acted on by both gravitational and EM fields.

The kinetic formulation is intrinsically asymptotic. This means that the theory (in particular the GK theory formulated in the next section) is characterized by a suitable species-dependent dimensionless physical parameter $\varepsilon_{M,s} \equiv \frac{r_{Ls}}{L} \ll 1$, where $s = i, e$ denotes the species index. Here $r_{Ls} = v_{\perp ths}/\Omega_{cs}$ is the species average Larmor radius, with $v_{\perp ths} = \{T_{\perp s}/M_s\}^{1/2}$ denoting the species thermal velocity perpendicular to the magnetic field and $\Omega_{cs} = Z_s eB/M_s c$ being the species Larmor frequency. Moreover, L is the characteristic length-scale of the spatial inhomogeneities of the EM field, defined as $L \sim L_B \sim L_E$, where L_B and L_E are the characteristic magnitudes of the gradients of the absolute values of the magnetic field $\mathbf{B}(\mathbf{x}, t)$ and the electric field $\mathbf{E}(\mathbf{x}, t)$, defined as $\frac{1}{L_B} \equiv \max\left\{\left|\frac{\partial}{\partial r_i} \ln B\right|, i = 1, 3\right\}$ and $\frac{1}{L_E} \equiv \max\left\{\left|\frac{\partial}{\partial r_i} \ln E\right|, i = 1, 3\right\}$, where the vector $\mathbf{x}$ denotes $\mathbf{x} = (R, z)$. Then, in analogy with Paper I, we define a unique parameter $\varepsilon_M \equiv \max\{\varepsilon_{M,s}, s = i, e\}$. For temperatures and magnetic fields typical of AD plasmas, we have $0 < \varepsilon_M \ll 1$.

In the following we will focus on solutions for the equilibrium magnetic field $\mathbf{B}$ which admit, at least locally, a family of nested and *open* axisymmetric toroidal magnetic surfaces $\{\psi(\mathbf{x})\} \equiv \{\psi(\mathbf{x}) = const.\}$, where ψ denotes
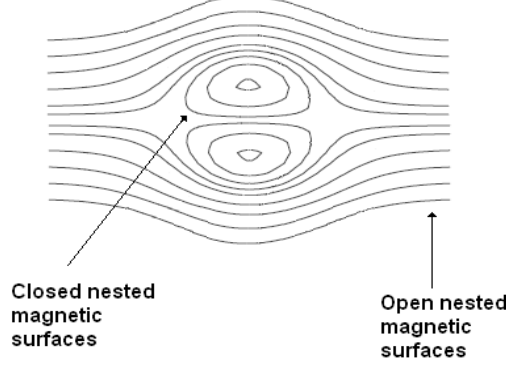
FIG. 1: Schematic comparison between the configuration of locally closed magnetic surfaces considered in Paper I and the case of open magnetic surfaces analysed in the present study.

the poloidal magnetic flux of $\mathbf{B}$. See Fig. 1 for a schematic comparison between the configuration of locally closed magnetic surfaces considered in Paper I and the case of open magnetic surfaces analyzed in the present study. A set of magnetic coordinates $(\psi, \varphi, \vartheta)$ can be defined locally, where ϑ is a curvilinear angle-like coordinate on the magnetic surfaces $\psi(\mathbf{x}) = const$. Each relevant physical quantity $G(\mathbf{x}, t)$ can then be conveniently expressed either in terms of the cylindrical coordinates or as a function of the magnetic coordinates, i.e. $G(\mathbf{x}, t) = \overline{G}(\psi, \vartheta, t)$, where the φ dependence has been suppressed due to the axisymmetry.

We require the EM field to be slowly varying in time, i.e., to be of the form

$$\left[\mathbf{E}(\mathbf{x}, \varepsilon_M^k t), \mathbf{B}(\mathbf{x}, \varepsilon_M^k t) \right], \tag{1}$$

with $k \geq 1$ being a suitable integer. This time dependence is connected with either external sources or boundary conditions for the KDF. In particular, we shall assume that the magnetic field is of the form

$$\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}^{self}(\mathbf{x}, \varepsilon_M^k t) + \mathbf{B}^{ext}(\mathbf{x}, \varepsilon_M^k t), \tag{2}$$

where $\mathbf{B}^{self}$ and $\mathbf{B}^{ext}$ denote the self-generated magnetic field produced by the AD plasma and a finite external magnetic field produced by the central object (in the case of neutron stars or white dwarfs). For greater generality, we shall not prescribe any relative orderings between the various components of the total magnetic field, which are taken to be of the form

$$\mathbf{B}^{self} = I(\mathbf{x}, \varepsilon_M^k t) \nabla \varphi + \nabla \psi_p(\mathbf{x}, \varepsilon_M^k t) \times \nabla \varphi, \tag{3}$$

$$\mathbf{B}^{ext} = \nabla \psi_D(\mathbf{x}, \varepsilon_M^k t) \times \nabla \varphi. \tag{4}$$

In particular, here $\mathbf{B}_T \equiv I(\mathbf{x}, \varepsilon_M^k t) \nabla \varphi$ and $\mathbf{B}_P \equiv \nabla \psi_p(\mathbf{x}, \varepsilon_M^k t) \times \nabla \varphi$ are the toroidal and poloidal components of the self-field, while the external magnetic field $\mathbf{B}^{ext}$ has to be purely poloidal, as a consequence of the axisymmetry, and is defined in terms of the vacuum potential $\psi_D(\mathbf{x}, \varepsilon_M^k t)$. As a consequence, the magnetic field can also be written in the equivalent form

$$\mathbf{B} = I(\mathbf{x}, \varepsilon_M^k t) \nabla \varphi + \nabla \psi(\mathbf{x}, \varepsilon_M^k t) \times \nabla \varphi, \tag{5}$$

where the function $\psi(\mathbf{x}, \varepsilon_M^k t)$ is defined as $\psi(\mathbf{x}, \varepsilon_M^k t) \equiv \psi_p(\mathbf{x}, \varepsilon_M^k t) + \psi_D(\mathbf{x}, \varepsilon_M^k t)$, with $k \geq 1$ and $(\psi, \varphi, \vartheta)$ defining a set of local magnetic coordinates (as implied by the equation $\mathbf{B} \cdot \nabla \psi = 0$ which is identically satisfied). Also, it is assumed that the charged particles of the plasma are subject to the action of *effective EM potentials* $\left\{ \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t), \mathbf{A}(\mathbf{x}, \varepsilon_M^k t) \right\}$, where $\mathbf{A}(\mathbf{x}, \varepsilon_M^k t)$ is the vector potential corresponding to the magnetic field of Eq.(5), while $\Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t)$ is given by

$$\Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t) = \Phi(\mathbf{x}, \varepsilon_M^k t) + \frac{M_s}{Z_s e} \Phi_G(\mathbf{x}, \varepsilon_M^k t), \tag{6}$$

with $\Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t)$, $\Phi(\mathbf{x}, \varepsilon_M^k t)$ and $\Phi_G(\mathbf{x}, \varepsilon_M^k t)$ denoting the *effective* electrostatic potential and the electrostatic and generalized gravitational potentials (the latter, in principle, being produced both by the central object and the

accretion disk). The effective electric field $\mathbf{E}_s^{eff}$ can then be defined as

$$\mathbf{E}_s^{eff} \equiv -\nabla \Phi_s^{eff} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \tag{7}$$

III.   GK THEORY FOR MAGNETIZED ACCRETION DISK PLASMAS

In this section we recall the GK theory appropriate for the description of AD plasmas. Its formulation is in fact a prerequisite for the construction of the kinetic quasi-stationary equilibria to be developed later. The appropriate generalization of GK theory allowing for the presence of strong gravitational fields should in principle be based on a covariant formulation [see [23–26]]. However, for non-relativistic plasmas within a gravitational field, the appropriate formulation can also be directly recovered via a suitable reformulation of the standard non-relativistic theory holding for magnetically confined plasmas [27–35].

In this case, the appropriate particle Lagrangian function can be represented in terms of the effective EM potentials $\left\{ \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t), \mathbf{A}(\mathbf{x}, \varepsilon_M^k t) \right\}$, with $k \geq 1$, where Φ_s^{eff} is defined in Eq.(6). In terms of the hybrid variables $\mathbf{z} \equiv (\mathbf{x}, \mathbf{v})$ (with $\mathbf{x}$ and $\mathbf{v}$ denoting respectively the particle position and velocity vectors), this is expressed as

$$\mathcal{L}_s(\mathbf{z}, \frac{d}{dt}\mathbf{z}, \varepsilon_M^k t) \equiv \dot{\mathbf{r}} \cdot \mathbf{P}_s - \mathcal{H}_s(\mathbf{z}, \varepsilon_M^k t), \tag{8}$$

where $\mathbf{P}_s \equiv \left[M_s \mathbf{v} + \frac{Z_s e}{c} \mathbf{A}(\mathbf{x}, \varepsilon_M^k t) \right]$ and

$$\mathcal{H}_s(\mathbf{z}, \varepsilon_M^k t) = \frac{M_s}{2} v^2 + Z_s e \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t) \tag{9}$$

denotes the corresponding Hamiltonian function in hybrid variables. The GK treatment for the Lagrangian (8) involves the construction - in terms of a perturbative expansion determined by means of a power series in ε_M - of a diffeomorphism of the form

$$\mathbf{z} \equiv (\mathbf{r}, \mathbf{v}) \rightarrow \mathbf{z}' \equiv (\mathbf{r}', \mathbf{v}'), \tag{10}$$

referred to as the *GK transformation*. Note that, in the following, we shall use a prime " $'$ " to denote a dynamical variable defined at the *guiding-center position* $\mathbf{r}'$ (or $\mathbf{x}'$ in axisymmetry). Here, by definition, the transformed variables $\mathbf{z}'$ (*GK state*) are constructed so that their time derivatives to the relevant order in ε_M have at least one ignorable coordinate (a suitably-defined gyrophase ϕ'). As an illustration, we show the formulation of the perturbative theory to leading-order in ε_M. In this case the GK transformation becomes simply

$$\begin{cases} \mathbf{r} = \mathbf{r}' - \frac{\mathbf{w}' \times \mathbf{b}'}{\Omega_{cs}'}, \\ \mathbf{v} = u'\mathbf{b}' + \mathbf{w}' + \mathbf{V}_{eff}', \end{cases} \tag{11}$$

where $\mathbf{w}' = w'\cos\phi' \mathbf{e}_1' + w'\sin\phi' \mathbf{e}_2'$, with ϕ' denoting the gyrophase angle. In the following, the GK transformation will be performed on all phase-space variables $\mathbf{z} \equiv (\mathbf{r}, \mathbf{v})$, *except* for the azimuthal angle φ which is left unchanged [36] and is therefore to be considered as one of the GK variables. Here $\mathbf{b}' = \mathbf{b}(\mathbf{x}', \varepsilon_M^k t)$, with $\mathbf{b}(\mathbf{x}, \varepsilon_M^k t) \equiv \mathbf{B}(\mathbf{x}, \varepsilon_M^k t)/B(\mathbf{x}, \varepsilon_M^k t)$, while $\Omega_{cs}' = \frac{Z_s e B'}{M_s c}$ and $\mathbf{V}_{eff}'$ are respectively the guiding-center Larmor frequency and the *effective drift velocity* produced by $\mathbf{E}_s^{'eff}$, namely

$$\mathbf{V}_{eff}'(\mathbf{x}, \varepsilon_M^k t) \equiv \frac{c}{B'} \mathbf{E}_s^{'eff} \times \mathbf{b}'. \tag{12}$$

The rest of the notation is standard, with u' and $\mathbf{w}'$ denoting respectively the parallel and perpendicular (guiding-center) velocities, both defined relative to the frame locally moving with velocity $\mathbf{V}_{eff}'$. It follows that, when expressed in terms of the GK variables $\mathbf{z}'$, the GK Lagrangian and Hamiltonian functions, $\mathcal{L}_s'$ and $\mathcal{H}_s'$, can be evaluated with the desired order of accuracy. In particular, to leading-order, i.e. neglecting corrections of $O(\varepsilon_M^n)$ with $n \geq 1$, $\mathcal{L}_s' = \mathcal{L}_s'^{(1)} + O(\varepsilon_M)$ and $\mathcal{H}_s' = \mathcal{H}_s'^{(1)} + O(\varepsilon_M)$, where $\mathcal{L}_s'^{(1)}$ and $\mathcal{H}_s'^{(1)}$ recover the customary expressions

$$\mathcal{L}_s'^{(1)} \equiv \dot{\mathbf{r}}' \cdot \frac{Z_s e}{c} \mathbf{A}_s^{'*} - \frac{\dot{\phi}'}{\Omega_{cs}'} m_s' B' - \mathcal{H}_s'^{(1)}, \tag{13}$$

$$\mathcal{H}_s'^{(1)} \equiv m_s' B' + \frac{M_s}{2} \left(u' \mathbf{b}' + \mathbf{V}_{eff}' \right)^2 + Z_s e \Phi_s'^*, \tag{14}$$

with the magnetic moment $m_s' \cong \mu_s' \equiv \frac{M_s w'^2}{2B'}$ to leading order, while the gyrophase-independent *modified EM potentials* $\left(\Phi_s'^*, \mathbf{A}_s'^* \right)$ are

$$\Phi_s'^* \cong \Phi_s'^{eff}, \tag{15}$$

$$\mathbf{A}_s'^* \cong \mathbf{A}' + \frac{M_s c}{Z_s e} \left(u' \mathbf{b}' + \mathbf{V}_{eff}' \right), \tag{16}$$

in the same approximation. It is important to stress that the GK theory can be performed in principle to arbitrary order in ε_M [27–35], thus permitting the explicit determination of m_s' and the modified EM potentials as well as the relevant guiding-center canonical momenta.

A. First integrals of motion and guiding-center adiabatic invariants for AD plasmas

The exact integrals of motion and the relevant adiabatic invariants corresponding respectively to Eqs.(8) and (13) can be immediately recovered. By definition, an adiabatic invariant P of order n with respect to ε_M is conserved only in an asymptotic sense, i.e., in the sense that $\frac{1}{\Omega_{cs}'} \frac{d}{dt} \ln P = 0 + O(\varepsilon_M^{n+1})$, where $n \geq 0$ is a suitable integer. First we notice that, under the assumptions of axisymmetry and of Eq.(1), the only first integral of motion is the canonical momentum $p_{\varphi s} \equiv \frac{\partial \mathcal{L}_s}{\partial \dot{\varphi}}$ conjugate to the ignorable azimuthal angle φ:

$$p_{\varphi s} = M_s R \mathbf{v} \cdot \mathbf{e}_\varphi + \frac{Z_s e}{c} \psi \equiv \frac{Z_s e}{c} \psi_{*s}. \tag{17}$$

Since the azimuthal angle φ is ignorable also for the GK Lagrangian $\mathcal{L}_s'$, it follows that the quantity $p_{\varphi s}' \equiv \frac{\partial \mathcal{L}_s'}{\partial \dot{\varphi}}$ is an adiabatic invariant of the prescribed order, according to the accuracy of the GK transformation used to evaluate $\mathcal{L}_s'$. We shall refer to $p_{\varphi s}'$ as the *guiding-center canonical momentum*. In particular, correct to $O(\varepsilon_M^k)$, with $k \geq 1$, one obtains

$$p_{\varphi s}' \equiv \frac{M_s}{B'} \left(u' I' - \frac{c \nabla' \psi' \cdot \nabla' \Phi_s'^{eff}}{B'} \right) + \frac{Z_s e}{c} \psi', \tag{18}$$

which is an adiabatic invariant of $O(\varepsilon_M^{k+1})$, with $k \geq 1$. Furthermore, the total particle energy

$$E_s = \frac{M_s}{2} v^2 + Z_s e \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^n t), \tag{19}$$

with $n \geq 1$, and the GK Hamiltonian $\mathcal{H}_s'$ are also adiabatic invariants of order n. Finally, in GK theory, by construction, the momentum $p_{\phi' s}' = \partial \mathcal{L}_s' / \partial \dot{\phi}'$ conjugate to the gyrophase, as well as the related magnetic moment m_s' defined as $m_s' \equiv \frac{Z_s e}{M_s c} p_{\phi' s}'$, are adiabatic invariants. As shown by Kruskal (1962 [37]) it is always possible to determine $\mathcal{L}_s'$ so that m_s' is an adiabatic invariant of arbitrary order in ε_M. In particular, the leading-order approximation is $m_s' \cong \mu_s' \equiv \frac{M_s w'^2}{2B'}$.

Note that the allowance of slow time variations for E_s is an elementary consequence of assumption (1), which allows us to describe realistic configurations of AD plasmas which slowly evolve in time.

B. Particle trapping phenomena

GK theory permits explicit treatment of particle trapping corresponding to the existence of forbidden regions for the motion of charged particles arising from conservation of energy and magnetic moment. Conservation of the guiding-center Hamiltonian (14) and the magnetic moment μ_s' (leading-order approximation) give rise to some implications. Combining the two identities to express the parallel velocity u', and using the definition (12), we find

$$u' = \pm \sqrt{\frac{2}{M_s} \left[\mathcal{H}_s'^{(1)} - \mu_s' B' - Z_s e \Phi_s'^{eff} - \frac{M_s}{2} V_{eff}'^2 \right]}. \tag{20}$$

Therefore u' is a local function of the guiding-center position vector $\mathbf{x}'$ and, thanks to axisymmetry, of the corresponding flux coordinates (ψ', ϑ'). Since the argument of the square root must be non-negative, this means that u' is only defined in the subset of the configuration space spanned by (ψ', ϑ') where this property holds. It follows that if the argument becomes null for given $\mathcal{H}_s'^{(1)}$ and μ_s', the parallel velocity must change sign so that the particle undergoes a spatial reflection. The points of the configuration space where this occurs are the so-called *mirror points*. The existence of these points may generate various kinetic phenomena in AD plasmas. In particular, for open magnetic surfaces, particles can in principle experience zero, one or two reflections corresponding respectively to *passing particles (PPs), bouncing particles (BPs)* and *trapped particles (TPs)*. In the present case, since the right hand side of Eq.(20) depends on the magnitude of the magnetic field (B'), the effective potential energy ($Z_s e \Phi_s'^{eff}$) and the centrifugal potential ($\frac{M_s}{2} V_{eff}'^2$), we shall refer to the TPs case as *gravitational EM trapping*. In Section VII we shall investigate some consequences of trapping phenomena for the dynamics of ADs.

IV. CONSTRUCTION OF THE QSA-KDF: GENERALIZED SOLUTION

In this section we show that the equilibrium generalized bi-Maxwellian solution for the KDF obtained in Paper I can be extended to QSA-KDFs describing axisymmetric AD plasmas with the following features:

1) The KDF is also axisymmetric;

2) Each species in the collisionless plasma is considered to be associated with a suitable set of *sub-species* (referring to the different populations mentioned above), each one having a different KDF;

3) Temperature anisotropy: for all of the species, it is assumed that different parallel and perpendicular temperatures are allowed (with respect to the local direction of the magnetic field);

4) Accretion flow velocity: a non-vanishing species dependent poloidal flow velocity is prescribed;

5) Open, locally nested magnetic flux surfaces: the magnetic field is taken to allow quasi-stationary solutions with magnetic flux lines belonging to open and locally nested magnetic surfaces;

6) Kinetic constraints: suitable functional dependencies are imposed so that the KDF is an adiabatic invariant;

7) Analytic form: the solution is required to be asymptotically "close" to a local bi-Maxwellian in order to permit comparisons with previous literature dealing with Maxwellian or a bi-Maxwellian KDFs (see for example [7, 9, 38]).

Requirement 2) is suggested by observations of collisionless plasmas. For example, in the solar wind plasma both ion and electron species are described by superpositions of shifted bi-Maxwellian distributions. Requirements 1) - 7) clearly imply that the solution cannot generally be a Maxwellian. However, in analogy with Paper I, it is possible to show that they can be fulfilled by a suitable modified bi-Maxwellian expressed solely in terms of first integrals of motion and adiabatic invariants [6, 10, 39]. It follows that this is necessarily a QSA-KDF. A set of fluid equations can then readily be determined using this solution, expressed in terms of four moments of the KDF [corresponding to the species number density, flow velocity and the parallel and perpendicular temperatures]. These equations which, by construction, satisfy a kinetic closure condition, are also useful for comparing with previous fluid treatments.

For consistency with the notation of Paper I, we again use the symbol "$\wedge$" to denote physical quantities which refer to the treatment of anisotropic temperatures, unless otherwise specified, but in the present work, for greater generality, the symbol "$*$" is used to denote variables which depend on both the canonical momentum ψ_{*s} and the total particle energy E_s.

In line with all of the previous requirements, it is possible to show that a particular solution for the QSA-KDF is given by:

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}}}{(2\pi/M_s)^{3/2}\left(T_{\|*s}\right)^{1/2}} \tag{21}$$
$$\times \exp\left\{-\frac{K_{*s}}{T_{\|*s}} - m_s'\widehat{\alpha_{*s}}\right\},$$

which we refer to as the *Generalized bi-Maxwellian KDF with parallel velocity perturbations*. Here $\widehat{f_{*s}}$ is defined in the phase-space $\Gamma = \Gamma_r \times \Gamma_u$, where Γ_r and Γ_u are both identified with suitable subsets of the Euclidean space $\mathbb{R}^3$. The notation is as follows:

$$\widehat{\beta_{*s}} \equiv \frac{\eta_s}{\widehat{T}_{\perp s}}, \tag{22}$$

$$\widehat{\alpha_{*s}} \equiv \frac{B'}{\widehat{\Delta_{T_s}}}, \tag{23}$$

$$K_{*s} \equiv E_s - \ell_{\varphi s}\varpi_{*s}, \tag{24}$$

with E_s and ψ_{*s} given by Eqs.(19) and (17) respectively, while $\frac{1}{\widehat{\Delta_{T_s}}} \equiv \frac{1}{\widehat{T}_{\perp s}} - \frac{1}{T_{\parallel *s}}$. By construction, $\ell_{\varphi s}$ has the dimensions of an angular momentum, while ϖ_{*s} has those of a frequency. In contrast with the solution obtained in Paper I, ϖ_{*s} is not necessarily associated here with a purely azimuthal leading-order velocity. In general K_{*s} can, in fact, be represented as

$$K_{*s} = E_s - \frac{Z_s e}{c} \psi_{*s} \Omega_{*s} - p'_{\varphi s} \xi_{*s} = H_{*s} - p'_{\varphi s} \xi_{*s}. \tag{25}$$

Here $H_{*s} \equiv E_s - \frac{Z_s e}{c} \psi_{*s} \Omega_{*s}$ has the same meaning as the analogous quantity used in Paper I, with Ω_{*s} being related to the azimuthal rotational frequency. In Eq.(25) ξ_{*s} is a frequency associated with the leading-order guiding-center canonical momentum $p'_{\varphi s}$ defined in Eq.(18), which is an adiabatic invariant depending on u' and, by definition, is independent of the gyrophase angle. As we shall show at the end of this section, this feature can be used to require that the QSA-KDF carries a non-vanishing parallel flow velocity. This can be related to a net accretion flow arising in the AD plasma. Finally, by substituting Eq.(25) into Eq.(21) we reach the equivalent representation for the QSA-KDF:

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}}}{(2\pi/M_s)^{3/2} \left(T_{\parallel *s}\right)^{1/2}} \tag{26}$$
$$\times \exp\left\{ -\frac{H_{*s}}{T_{\parallel *s}} + \frac{p'_{\varphi s} \xi_{*s}}{T_{\parallel *s}} - m'_s \widehat{\alpha_{*s}} \right\}.$$

In order for the solution (26) [or equivalently (21)] to be a function of the integrals of motion and of the adiabatic invariants, the functions $\{\Lambda_{*s}\} \equiv \left\{ \widehat{\beta_{*s}}, \widehat{\alpha_{*s}}, T_{\parallel *s}, \Omega_{*s}, \xi_{*s} \right\}$, which we will refer to as *structure functions*, must be adiabatic invariants by themselves. To further generalize the solution of Paper I, we shall here retain a functional dependence on both the total particle energy and the canonical momentum, thus imposing the functional dependencies

$$\Lambda_{*s} = \Lambda_{*s}\left(\psi_{*s}, E_s\right), \tag{27}$$

which will be referred to in the following as *kinetic constraints*. The kinetic constraints (27) provide the most general solution for $\widehat{f_{*s}}$. It can be shown that the physical motivation behind imposing these dependencies lies essentially in the fact that the asymptotic condition of small inverse aspect ratio (adopted previously in Paper I) is no longer valid. In the present context, the kinetic solution is no longer restricted to localized spatial domains in the disk but applies to the general configuration of open magnetic surfaces. This in turn implies that the structure functions are generally not simply flux-functions on the magnetic surfaces. In previous treatments (Paper I and Ref.[36]), the structure functions were identified with $\left\{ \widehat{\beta_{*s}}, \widehat{\alpha_{*s}}, T_{\parallel *s}, \Omega_{*s} \right\}$ and $\left\{ \widehat{\beta_{*s}} \equiv N_{*s}, T_{*s} \right\}$ for non-isotropic and isotropic generalized Maxwellian KDFs respectively.

Some basic properties of $\widehat{f_{*s}}$ are:

Property 1: $\widehat{f_{*s}}$ is itself an adiabatic invariant, and is therefore an asymptotic solution of the stationary Vlasov equation, i.e., a QSA-KDF;

Property 2: $\widehat{f_{*s}}$ is only defined in the subset of phase-space where the adiabatic invariants $p'_{\varphi s}$, $\mathcal{H}'^{(1)}_s$ and m'_s are defined. It follows that $\widehat{f_{*s}}$ is suitable for describing both circulating and trapped particles (see the related discussion in Section 7);

Property 3: all of the velocity-moment equations obtained from the Vlasov equation (and in particular the continuity and linear momentum fluid equations) are identically satisfied in an asymptotic sense, i.e., neglecting corrections of $O\left(\varepsilon_M^{n+1}\right)$;

Property 4: its velocity moments, to be identified with the fluid fields, are unique once $\widehat{f_{*s}}$ is prescribed in terms of the structure functions;

Property 5: it generalizes the solution earlier presented in Paper I: a) by using both $p'_{\varphi s}$ and m'_s as adiabatic invariants and b) because of the new kinetic constraints.

It follows immediately that the solution (26) does indeed carry finite parallel velocity perturbations. Invoking the definitions (18) and (25), Eq.(26) can be re-written as

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}} \exp\left[\frac{X_{*s}}{T_{\parallel *s}}\right]}{(2\pi/M_s)^{3/2} \left(T_{\parallel *s}\right)^{1/2}} \tag{28}$$
$$\times \exp\left\{ -\frac{M_s \left(\mathbf{v} - \mathbf{V}_{*s} - U'_{\parallel *s} \mathbf{b}'\right)^2}{2T_{\parallel *s}} - m'_s \widehat{\alpha_{*s}} \right\},$$

where $\mathbf{V}_{*s} = \mathbf{e}_\varphi R\Omega_{*s}(\psi_{*s}, E_s)$ and

$$X_{*s} \equiv M_s \frac{|\mathbf{V}_{*s}|^2}{2} + \frac{Z_s e}{c}\psi\Omega_{*s} - Z_s e\Phi_s^{eff} + \Upsilon'_{*s}. \tag{29}$$

Here the function Υ'_{*s} is defined as

$$\Upsilon'_{*s} \equiv \frac{M_s U_{\|*s}'^2}{2}\left(1 + \frac{2\Omega_{*s}}{\xi_{*s}}\right) + \tag{30}$$
$$- \left(\frac{M_s c\nabla'\psi' \cdot \nabla'\Phi_s'^{eff}}{B'^2} - \frac{Z_s e}{c}\psi'\right)\xi_{*s},$$

with $U_{\|*s}' = \frac{I'}{B'}\xi_{*s}(\psi_{*s}, E_s)$. Note that $U_{\|*s}'$ is non-zero only if the toroidal magnetic field is non-vanishing. This quantity is independent of $\mathbf{V}_{*s}$ and is clearly associated with a parallel flow velocity (i.e., having both poloidal and toroidal components), referred to here as a *parallel velocity perturbation*. This perturbation enters the solution via the adiabatic invariant $p'_{\varphi s}$ and therefore its inclusion is consistent with the requirement that KDF is an adiabatic invariant.

Finally we note that the same kinetic constraints (27) also apply to the solution (28). However, the functions $\widehat{\beta_{*s}}\exp\left[\frac{X_{*s}}{T_{\|*s}}\right]$, $\mathbf{V}_{*s}$, $U_{\|*s}'$ and $T_{\|*s}$ cannot be directly regarded as *fluid fields*, since they still depend on the single particle velocity via the canonical momentum ψ_{*s} and the particle energy E_s.

V. ANALYTICAL EXPANSION

Based on Properties 1-5, in this section we determine an approximate analytical expression for $\widehat{f_{*s}}$ obtained by means of suitable asymptotic expansions. These are carried out in terms of the following two dimensionless parameters:

1) ε_s: which is related to the canonical momentum ψ_{*s}. This is defined as (cf Paper I) $\varepsilon_s \equiv \left|\frac{L_{\varphi s}}{p_{\varphi s} - L_{\varphi s}}\right| = \left|\frac{M_s R v_\varphi}{\frac{Z_s e}{c}\psi}\right|$, where $v_\varphi \equiv \mathbf{v} \cdot \mathbf{e}_\varphi$ and $L_{\varphi s}$ denotes the species particle angular momentum. We refer to the AD plasma as being *strongly magnetized* if $0 < \varepsilon_s \ll 1$;

2) σ_s: which is related to the total particle energy E_s. This is defined as $\sigma_s \equiv \left|\frac{\frac{M_s}{2}v^2}{Z_s e\Phi_s^{eff}}\right|$, i.e., it is the ratio between the kinetic energy and potential energy of the particle. For bound orbits $E_s < 0$, and so $\sigma_s < 1$.

In the following, we treat ε_s and σ_s as infinitesimals of the same order, with $\varepsilon_s \sim \sigma_s \ll 1$ and then ε_s and σ_s can be used for performing a Taylor expansion of the implicit dependencies contained in the structure functions by setting $\psi_{*s} \cong \psi + O(\varepsilon_s)$ and $E_s \cong Z_s e\Phi_s^{eff} + O(\sigma_s)$ to leading order. This implies that the linear asymptotic expansion for the structure functions, obtained neglecting corrections of $O(\varepsilon_s\sigma_s)$, as well as of $O(\varepsilon_s^k)$ and $O(\sigma_s^k)$, with $k \geq 2$, is

$$\Lambda_{*s} \cong \Lambda_s + (\psi_{*s} - \psi)\left[\frac{\partial\Lambda_{*s}}{\partial\psi_{*s}}\right]_{\substack{\psi_{*s}=\psi \\ E_s=Z_s e\Phi_s^{eff}}} +$$
$$+ \left(E_s - Z_s e\Phi_s^{eff}\right)\left[\frac{\partial\Lambda_{*s}}{\partial E_s}\right]_{\substack{\psi_{*s}=\psi \\ E_s=Z_s e\Phi_s^{eff}}}, \tag{31}$$

where

$$\Lambda_s \equiv \Lambda_{*s}\Big|_{\substack{\psi_{*s}=\psi \\ E_s=Z_s e\Phi_s^{eff}}}. \tag{32}$$

To perform the corresponding expansion for $\widehat{f_{*s}}$, we leave unchanged the dependence in terms of the guiding-center canonical momentum $p'_{\varphi s}$, while retaining the leading-order approximation for the magnetic moment only in the linear perturbation terms of Eq.(31). Then, it is straightforward to prove that for strongly magnetized and bound plasmas, the following relation holds to leading-order:

$$\widehat{f_{*s}} \cong \widehat{f_s}\left(p'_{\varphi s}, m'_s\right)\left[1 + h_{Ds}^1 + h_{Ds}^2\right], \tag{33}$$

where h_{Ds}^1 and h_{Ds}^2 represent the so-called *diamagnetic parts* of $\widehat{f_{*s}}$ (see the definition below). The definitions are then as follows:

First, the leading-order distribution $\widehat{f}_s\left(p'_{\varphi s}, m'_s\right)$ is expressed as

$$\widehat{f}_s\left(p'_{\varphi s}, m'_s\right) = \frac{n_s}{\left(2\pi/M_s\right)^{3/2}\left(T_{\|s}\right)^{1/2} T_{\perp s}}$$
$$\times \exp\left\{-\frac{M_s\left(\mathbf{v} - \mathbf{V}_s - U'_{\|s}\mathbf{b}'\right)^2}{2T_{\|s}} - m'_s\frac{B'}{\Delta_{T_s}}\right\} \tag{34}$$

which we will here call the *bi-Maxwellian KDF with parallel velocity perturbations*. Here $\frac{1}{\Delta_{T_s}} \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{\|s}}$ is related to the temperature anisotropy, the number density is defined as

$$n_s = \eta_s \exp\left[\frac{X_s}{T_{\|s}}\right] \tag{35}$$

and

$$X_s \equiv \left(M_s\frac{R^2\Omega_s^2}{2} + \frac{Z_s e}{c}\psi\Omega_s - Z_s e\Phi_s^{eff} + \Upsilon'_s\right), \tag{36}$$

with η_s denoting the *pseudo-density*. The function Υ'_s is defined as

$$\Upsilon'_s \equiv \frac{M_s U'^2_{\|s}}{2}\left(1 + \frac{2\Omega_s}{\xi'_s}\right) +$$
$$- \left(\frac{M_s c\nabla'\psi' \cdot \nabla'\Phi_s'^{eff}}{B'^2} - \frac{Z_s e}{c}\psi'\right)\xi_s. \tag{37}$$

Note that $\mathbf{V}_s = \mathbf{e}_\varphi R\Omega_s$ and $U'_{\|s} = \frac{I'}{B'}\xi_s$ define, respectively, the leading-order azimuthal flow velocity and the leading-order parallel velocity perturbation of the fluid. Then, the following kinetic constraints are implied from (27), to leading-order, for the structure functions:

$$\Lambda_s = \Lambda_s\left(\psi, Z_s e\Phi_s^{eff}\right). \tag{38}$$

Second, the diamagnetic parts h_{Ds}^1 and h_{Ds}^2 of $\widehat{f}_{*s}$, due respectively to the expansions of the canonical momentum and the total energy, are given by

$$h_{Ds}^1 = \left\{\frac{cM_s R}{Z_s e}\left[Y_1 + Y_3\right] + \frac{M_s R}{T_{\|s}}Y_2\right\}\left(\mathbf{v}\cdot\widehat{e}_\varphi\right), \tag{39}$$

$$h_{Ds}^2 = \frac{M_s}{2Z_s e}\left\{Y_4 - \frac{Z_s e}{T_{\|s}}Y_5 + \frac{p'_{\varphi s}\xi_s}{T_{\|s}}C_{5s}\right\}v^2. \tag{40}$$

Here Y_i, $i = 1, 5$, is defined as

$$Y_1 \equiv \left[A_{1s} + A_{2s}\left(\frac{H_s}{T_{\|s}} - \frac{1}{2}\right) - \mu'_s\widehat{A_{4s}}\right], \tag{41}$$

$$Y_2 \equiv \Omega_s\left[1 + \psi A_{3s}\right], \tag{42}$$

$$Y_3 \equiv \left[\frac{p'_{\varphi s}\xi_s}{T_{\|s}}A_{5s} - A_{2s}\frac{p'_{\varphi s}\xi_s}{T_{\|s}}\right], \tag{43}$$

$$Y_4 \equiv \left[C_{1s} + C_{2s}\left(\frac{H_s}{T_{\|s}} - \frac{1}{2}\right) - \mu'_s\widehat{C_{4s}}\right], \tag{44}$$

$$Y_5 \equiv \left[1 + \frac{\Omega_s\psi}{c}C_{3s}\right], \tag{45}$$

where $H_s = E_s - \frac{Z_s e}{c}\psi_s\Omega_s$ and the following definitions have been introduced: $A_{1s} \equiv \frac{\partial \ln\beta_s}{\partial\psi}$, $A_{2s} \equiv \frac{\partial \ln T_{\|s}}{\partial\psi}$, $A_{3s} \equiv \frac{\partial \ln\Omega_s}{\partial\psi}$, $\widehat{A_{4s}} \equiv \frac{\partial\widehat{\alpha_s}}{\partial\psi}$, $A_{5s} \equiv \frac{\partial \ln\xi_s}{\partial\psi}$ and $C_{1s} \equiv \frac{\partial \ln\beta_s}{\partial\Phi_s^{eff}}$, $C_{2s} \equiv \frac{\partial \ln T_{\|s}}{\partial\Phi_s^{eff}}$, $C_{3s} \equiv \frac{\partial \ln\Omega_s}{\partial\Phi_s^{eff}}$, $\widehat{C_{4s}} \equiv \frac{\partial\widehat{\alpha_s}}{\partial\Phi_s^{eff}}$, $C_{5s} \equiv \frac{\partial \ln\xi_s}{\partial\Phi_s^{eff}}$.

We should make a number of comments here:

1) The functional forms of the leading-order number density, the parallel and azimuthal flow velocities and the temperatures carried by the bi-Maxwellian KDF, are naturally determined in terms of ψ and $Z_s e \Phi_s^{eff}$. The effective potential Φ_s^{eff} is generally a function of the form $\Phi_s^{eff} = \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t)$, with $\mathbf{x} = (R, z)$, since generally neither the gravitational potential nor the electrostatic potential are expected to be flux functions in the present case. Hence, in magnetic coordinates, it follows that the structure functions are of the form $\Lambda_s \equiv \overline{\Lambda_s}\left(\psi, \vartheta, \varepsilon_M^k t\right)$;

2) The coefficients A_{is} and C_{is}, $i = 1, 5$, can be identified with effective *thermodynamic forces*: A_{5s} carries the contribution of the parallel velocity perturbation, while the C_{is}, $i = 1, 5$, are due to the energy dependence contained in the structure functions;

3) We stress that the energy dependence contained in the kinetic constraints is non trivial and cannot be included simply by redefining the structure functions (e.g., by transforming the magnetic coordinates). In fact, besides modifying the leading order structure functions (see point 1 above), it gives rise to the new diamagnetic contribution h_{Ds}^2. Eq.(33) is therefore a generalization of the analogous solution obtained in Paper I, which also appears in standard tokamak transport theory [36], where the relevant structure functions were considered solely as flux functions. Including the effect of the parallel velocity perturbations gives rise to contributions to h_{Ds}^2 which are even with respect to u';

4) In the analytical expansion, we have assumed that the scale-length L is of the same order in ε_s as the characteristic scale-lengths associated with the structure functions;

5) We have performed the analysis distinguishing between the different plasma species. Since this is an asymptotic estimation, the analytical expansion can be different for ions and electrons, particularly for the terms appearing in the diamagnetic part, depending on the relative magnitudes of the parameters ε_s and σ_s. On the other hand, because of the double expansion and the energy dependence, the asymptotic solution for the two species can hold also in different spatial domains;

6) The KDF $\widehat{f}_s\left(p'_{\varphi s}, m'_s\right)$ also satisfies Property 2: namely, it is only defined in the subset of phase-space where the parallel velocity $|u'|$ is a real function. It is therefore suitable for properly describing particle trapping;

7) Finally, we stress that the QSA-KDF (26) obtained here, reduces asymptotically to the expression reported in previous paper (see Eq.(10) in Paper I) when the following conditions are satisfied: a) parallel velocity perturbations are ignored, namely the structure function ξ_{*s} is set to zero; b) closed nested magnetic surfaces are considered; c) large aspect ratio ordering, $1/\delta \gg 1$, is invoked (see the definition in Paper I). In this case, the effective potential is solely a flux-function to leading order, while the diamagnetic contribution h_{Ds}^2 can be shown to be of higher order than h_{Ds}^1.

VI. MOMENT EQUATIONS

In this section we discuss the connection between the kinetic treatment presented here and the corresponding fluid approach, obtained by describing the plasma in terms of a suitable set of fluid fields. The latter can in principle be specified as required by experimental observations and identified with the relevant physical observables. Important practical aspects of the present theory concern the explicit evaluation of the fluid fields associated with the QSA-KDF, and the conditions for validity of the relevant moment equations.

For definiteness, let us require that:

1. The KDF, the EM fields $\{\mathbf{E}, \mathbf{B}\}$ and the corresponding EM potentials $\{\Phi, \mathbf{A}\}$ are all exactly axisymmetric and, moreover, stationary in an asymptotic sense, i.e. neglecting corrections of $O(\varepsilon_M^{n+1})$;

2. The KDF is identified with the QSA-KDF $\widehat{f_{*s}}\left(E_s, \psi_{*s}, m'_s\right)$ which, by assumption, is required to be an adiabatic invariant of $O(\varepsilon_M^{n+1})$. By construction $\widehat{f_{*s}}\left(E_s, \psi_{*s}, m'_s\right)$ is a solution of the *asymptotic Vlasov equation*

$$\frac{1}{\Omega'_{cs}} \frac{d}{dt} \ln \widehat{f_{*s}} = 0 + O\left(\varepsilon_M^{n+1}\right). \tag{46}$$

This equation holds by definition up to infinitesimals of $O\left(\varepsilon_M^{n+1}\right)$, where n is an arbitrary positive integer;

3. The magnetic field is taken to be of the form (5).

As a basic consequence of these assumptions, the stationary fluid equations following from the Vlasov equation are necessarily all identically satisfied in an asymptotic sense, i.e., again neglecting corrections of $O\left(\varepsilon_M^{n+1}\right)$. In fact if

$Z(\mathbf{x})$ is an arbitrary weight function, identified for example with $Z = (1, \mathbf{v}, v^2)$, then the generic moment of Eq.(46) is:

$$\int_{\Gamma_u} d^3 v Z \frac{d}{dt} \widehat{f_{*s}} = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{47}$$

where Γ_u denotes the appropriate velocity space of integration. Using the chain rule, this can be written as

$$\int_{\Gamma_u} d^3 v Z \left\{ \frac{d\psi_*}{dt} \frac{\partial \widehat{f_{*s}}}{\partial \psi_*} + \frac{dE_s}{dt} \frac{\partial \widehat{f_{*s}}}{\partial E_s} + \frac{dm_s'}{dt} \frac{\partial \widehat{f_{*s}}}{\partial m_s'} \right\} =$$
$$= 0 + O\left(\varepsilon_M^{n+1}\right). \tag{48}$$

On the other hand, Eq.(47) can also be represented as

$$\int_{\Gamma_u} d^3 v \left\{ \frac{d}{dt} \left[Z \widehat{f_{*s}} \right] - \widehat{f_{*s}} \frac{d}{dt} Z \right\} = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{49}$$

which recovers the usual form of the velocity-moment equations in terms of suitable (and *uniquely defined*) fluid fields. For $Z = (1, \mathbf{v})$ one obtains, in particular, that the species continuity and linear momentum fluid equations are satisfied identically up to infinitesimals of $O\left(\varepsilon_M^{n+1}\right)$:

$$\nabla \cdot \left(n_s^{tot} \mathbf{V}_s^{tot} \right) = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{50}$$

$$M_s \mathbf{V}_s^{tot} \cdot \nabla \mathbf{V}_s^{tot} + \nabla \cdot \underline{\underline{\Pi}}_s^{tot} + Z_s e n_s^{tot} \nabla \Phi_s^{eff} +$$
$$- \frac{Z_s e}{c} \mathbf{V}_s^{tot} \times \mathbf{B} = 0 + O\left(\varepsilon_M^{n+1}\right). \tag{51}$$

Similarly, the law of conservation of the species total canonical momentum can be recovered by setting $Z = \psi_{*s}$, namely

$$\int_{\Gamma_u} d^3 v \frac{d}{dt} \left[\psi_{*s} \widehat{f_{*s}} \right] = 0 + O\left(\varepsilon_M^{n+1}\right). \tag{52}$$

In the stationary case this implies the *species angular momentum conservation law*

$$\nabla \cdot \left[R^2 \underline{\underline{\Pi}}_s^{tot} \cdot \nabla \varphi + \mathbf{V}_s^{tot} L_s^{tot} \right] + \frac{Z_s e}{c} \nabla \psi \cdot n_s^{tot} \mathbf{V}_s^{tot} = 0 \tag{53}$$

for the species angular momentum

$$L_s^{tot} \equiv M_s R^2 n_s^{tot} \mathbf{V}_s^{tot} \cdot \nabla \varphi. \tag{54}$$

Here the notation is standard. In particular the following velocity moments of the QSA-KDF can be introduced:
    a) *species number density*

$$n_s^{tot} \equiv \int_{\Gamma_u} d^3 v \widehat{f_{*s}}; \tag{55}$$

    b) *species flow velocity*

$$\mathbf{V}_s^{tot} \equiv \frac{1}{n_s^{tot}} \int_{\Gamma_u} d^3 v \mathbf{v} \widehat{f_{*s}}; \tag{56}$$

    c) *species tensor pressure*

$$\underline{\underline{\Pi}}_s^{tot} \equiv \int_{\Gamma_u} d^3 v M_s \left(\mathbf{v} - \mathbf{V}_s^{tot} \right) \left(\mathbf{v} - \mathbf{V}_s^{tot} \right) \widehat{f_{*s}}; \tag{57}$$

d) *species canonical toroidal momentum*

$$L_{cs}^{tot} \equiv \int_{\Gamma_u} d^3v \frac{Z_s e}{c} \psi_{*s} \widehat{f_{*s}}. \tag{58}$$

It is worth remarking here that *the velocity moments are unique once the QSA-KDF $\widehat{f_{*s}}$ [see Eq.(21)] is prescribed in terms of the structure functions Λ_{*s}*. On the other hand, as a result of Eqs.(46) and (47), it follows that the stationary fluid moments calculated in terms of the QSA-KDF $\widehat{f_{*s}}$ are identically solutions of the corresponding stationary fluid moment equations. In particular, imposing the *quasi-neutrality condition* in the sense

$$\sum_{s=i,e} Z_s e n_s^{tot} = 0 + O\left(\varepsilon_M^k\right) \tag{59}$$

with $k \geq 2$, the total fluid canonical toroidal momentum and the fluid angular momentum necessarily coincide, namely

$$L^{tot} \equiv \sum_{s=i,e} L_s^{tot} \equiv \sum_{s=i,e} L_{cs}^{tot}. \tag{60}$$

Let us now illustrate explicitly how it is possible to carry out such a calculation within the present theory. The evaluation of the previous fluid fields can be made by using the asymptotic analytical solution of the QSA-KDF $\widehat{f_{*s}}$ derived in the previous section and given by Eq.(33). For example, adopting this expansion in the limit of strongly magnetized plasmas, from Eq.(55) the species number density becomes

$$n_s^{tot} \cong \int_{\Gamma_u} d^3v \left\{ \widehat{f_s} \left[1 + h_{Ds}^1 + h_{Ds}^2 \right] \right\}, \tag{61}$$

in which the diamagnetic corrections to the bi-Maxwellian KDF $\widehat{f_s}$ are polynomial functions of the particle velocity. Analogous expressions can also be obtained in a straightforward way for the remaining fluid moments. As pointed out in Paper I and subsequently in Ref.[16], the expansion procedure for $\widehat{f_{*s}}$ can in principle be performed to higher order, allowing for the analytical computation of the corresponding quasi-stationary fluid fields and the determination of the relevant kinetic closure conditions for the stationary moment equations. In the present context we stress that the theory allows the treatment of multiple-species plasmas including, in particular, particle trapping phenomena. This is taken into account by proper definition of the velocity sub-space Γ_u in which the integrations are performed. In fact, charged particles in both open and closed configurations can have mirror points (TPs and BPs) or be PPs, which are free to stream through the boundaries of the domain. These populations give different contributions to the relevant fluid fields and therefore require separate statistical treatments. The explicit calculation of fluid fields requires also a preliminary *inverse transformation* representing all quantities in terms of the actual particle positions (the FLR expansion, see Eq.(11)). This introduces further correction terms of order ε_M^k, $k \geq 1$, into the final analytical expressions. In contrast with the conclusion reached in Paper I, here we expect these FLR corrections to be non-negligible due to the requirement $\varepsilon_{M,s} \sim \varepsilon_s$ holding for open-field configurations.

VII. SLOW TIME-EVOLUTION OF THE AXISYMMETRIC QSA-KDF

In this section we investigate the temporal evolution of the axisymmetric QSA-KDF, consistent with the assumptions of Section 2 and the results of Sections 3 and 4. Two different issues must be addressed: giving an estimate of the maximum time interval over which the QSA-KDF can be regarded as an asymptotic stationary solution; and determining the solution of the Vlasov equation for time intervals longer than the equilibrium one.

For our explicit determination of the time evolution of the QSA-KDF, we make the following assumptions:

1) That the plasma can be treated as a continuous medium in the kinetic description. This requires that the species kinetic equation holds on time and spatial scales which are much longer than the corresponding Langmuir characteristic times and Debye lengths;

2) That we are considering timescales much shorter than the species characteristic collisional time τ_C, so that it is appropriate to use the Vlasov equation;

3) That the species KDF and the EM fields vary slowly in time and space with respect to the corresponding Larmor times and radii, so that the GK description is valid;

4) That the EM and gravitational fields vary slowly in time, in accordance with Eq.(1), so that the total energy E_s is an adiabatic invariant. In particular, we require:

$$\frac{d}{dt}E_s = Z_s e \frac{\partial}{\partial t}\Phi_s^{eff} - \frac{Z_s e}{c}\mathbf{v}\cdot\frac{\partial}{\partial t}\mathbf{A},\tag{62}$$

which implies that $\tau_{Ls}\frac{d}{dt}\ln E_s \sim O\left(\varepsilon_{M,s}^{n+1}\right)$, with $n \geq 0$. Consistently with the properties of solution (26), we take $n = 0$ as a specific case. Note that from here on, $\tau_{Ls} \equiv \frac{1}{\Omega_{cs}'}$ will denote the species characteristic time associated with the Larmor rotation (the *Larmor rotation time*). Since $\frac{d}{dt}\Omega_{*s} = \frac{dE_s}{dt}\frac{\partial}{\partial E_s}\Omega_{*s}$, it follows that

$$\frac{d}{dt}H_{*s} = \frac{d}{dt}E_s\left[1 - \frac{Z_s e}{c}\psi_{*s}\frac{\partial}{\partial E_s}\Omega_{*s}\right];\tag{63}$$

5) That the magnetic moment m_s' and the guiding-center canonical momentum $p_{\varphi s}'$ can be taken as adiabatic invariants of $O\left(\varepsilon_{M,s}^j\right)$, with $j \geq n$. The ordering $\tau_{Ls}\frac{d}{dt}\ln p_{\varphi s}' \sim O\left(\varepsilon_{M,s}^2\right)$ holds for the leading-order expression for $p_{\varphi s}'$ adopted here as follows from Eq.(18) and the fact that, by definition, higher-order correction terms, $\Delta p_{\varphi s}'$, to $p_{\varphi s}'$ are independent of the gyrophase angle ϕ'. In fact, denoting by $\mathcal{L}_s'^{(2)}$ the second-order GK Lagrangian, $\Delta p_{\varphi s}'$ can be estimated as $\Delta p_{\varphi s}' = \frac{\partial}{\partial\varphi}\left[\mathcal{L}_s'^{(2)} - \mathcal{L}_s'^{(1)}\right]$ where, by construction, $\mathcal{L}_s'^{(1)}$ and $\mathcal{L}_s'^{(2)}$ are both gyrophase independent. Note that the assumption made here requires the construction of a higher-order GK theory in order to correctly determine m_s' to the required order in the Larmor-radius expansion.

The time evolution of the QSA-KDF is in principle determined by two different mechanisms: the explicit time variation of the EM and gravitational fields, and the time variation of the guiding-center adiabatic invariants. However, the choice of the orderings in 4) and 5) above, allows the time dependence produced only by the EM and gravitational fields to be singled out.

When assumptions 1) - 5) above hold, it follows that $\tau_{Ls}\frac{d}{dt}\ln\widehat{f_{*s}} = 0 + O\left(\varepsilon_{M,s}^{n+1}\right)$, with $n \geq 0$ being determined by Eq.(62). Then, ignoring higher-order corrections

$$\frac{d}{dt}\ln\widehat{f_{*s}} = \frac{dE_s}{dt}S_s,\tag{64}$$

where

$$\begin{aligned}S_s \equiv{} & \frac{\partial\ln\widehat{\beta_{*s}}}{\partial E_s} - m_s'\frac{\partial\widehat{\alpha_{*s}}}{\partial E_s} + \frac{p_{\varphi s}'}{T_{\|*s}}\frac{\partial\xi_{*s}}{\partial E_s} + \\ & + \left(\frac{H_{*s}}{T_{\|*s}} - \frac{1}{2} + \frac{p_{\varphi s}'\xi_{*s}}{T_{\|*s}}\right)\frac{\partial\ln T_{\|*s}}{\partial E_s} + \\ & - \frac{1}{T_{\|*s}}\left(1 - \frac{Z_s e}{c}\psi_{*s}\frac{\partial\Omega_{*s}}{\partial E_s}\right),\end{aligned}\tag{65}$$

and so the solution $\widehat{f_{*s}}$ can be regarded as an exact kinetic equilibrium for all times $t \geq 0$ such that

$$\tau_{Ls} \ll t \ll t_{\sup} \ll \tau_C,\tag{66}$$

where $t_{\sup} \equiv \frac{\tau_{Ls}}{\varepsilon_{M,s}^{n+1}}$. Within the scope of the above assumptions, we now determine the dynamical evolution equation which describes the slow time-evolution of the QSA-KDF $\widehat{f_{*s}}$, for time intervals such that t is within

$$t_{\sup} \ll t \ll \tau_C.\tag{67}$$

In analogy with Ref.[36], we denote by

$$f_s \equiv \widehat{f_{*s}} + g_s'\tag{68}$$

the exact solution of the collisionless Vlasov equation, for which $\frac{d}{dt}f_s = 0$. Here g_s' is referred to as the *reduced KDF*. Following the discussion in Ref.[36], regarding the evaluation of $\frac{d}{dt}g_s'$: it is straightforward to prove that g_s' is gyrophase independent, to lowest order, in the sense that $\frac{\partial g_s'}{\partial\phi'} = 0$. Therefore, identifying the GK variables with the set
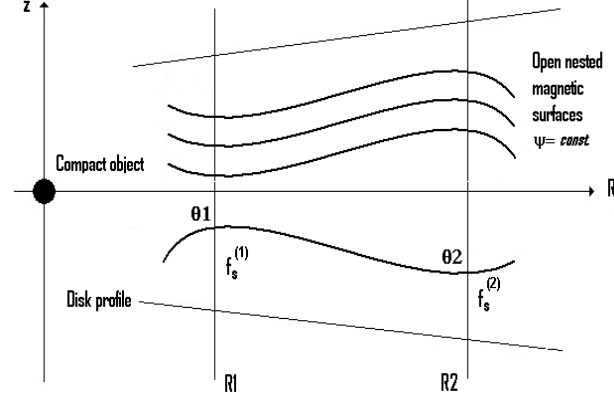
FIG. 2: Schematic view of the configuration geometry (not to scale) and meaning of the notation.

$\mathbf{z} \equiv \left(\vartheta', \varphi, p'_{\varphi s}, \mathcal{H}'^{(1)}_s, m'_s, \phi' \right)$, we shall assume that g'_s is axisymmetric and of the form $g'_s = g'_s \left(\vartheta', p'_{\varphi s}, \mathcal{H}'^{(1)}_s, m'_s, t \right)$. The gyro-averaged dynamical equation for g'_s can then be obtained to next order by introducing the gyro-average operator $\langle ... \rangle_{\phi'}$ defined as

$$\langle ... \rangle_{\phi'} \equiv \frac{1}{2\pi} \int_0^{2\pi} (...) \, d\phi', \tag{69}$$

with the operation being performed while all of the other GK variables are held fixed [36]. It follows that, to leading-order, $\frac{d}{dt} g'_s \cong \frac{\partial}{\partial t} g'_s + \dot{\vartheta}' \frac{\partial}{\partial \vartheta'} g'_s$, where the time variation of the guiding-center magnetic coordinate ϑ' is given by $\dot{\vartheta}' \cong \dot{\mathbf{r}}' \cdot \nabla' \vartheta' \cong \left[u' \mathbf{b}' + \mathbf{V}'_{eff} \right] \cdot \nabla' \vartheta'$ to leading-order, with the equation of motion for $\dot{\mathbf{r}}'$ following from the gyrokinetic Lagrangian [e.g. from the leading-order Eq.(13)]. Then, consistently with these assumptions and ignoring higher-order corrections, it is found that the GK reduced KDF g'_s obeys the *reduced GK-Vlasov equation*

$$\frac{\partial}{\partial t} g'_s + \dot{\vartheta}' \frac{\partial}{\partial \vartheta'} g'_s = - \left\langle \widehat{f_{*s}} S_s \frac{dE_s}{dt} \right\rangle_{\phi'}. \tag{70}$$

It follows that, to leading-order

$$\left\langle \widehat{f_{*s}} S_s \frac{dE_s}{dt} \right\rangle_{\phi'} \cong f'_s \left\langle S_s \frac{dE_s}{dt} \right\rangle_{\phi'}. \tag{71}$$

Denoting $\widehat{f_{*s}} \equiv F_s \left(\psi_{*s}, H_{*s}, p'_{\varphi s}, m'_s \right)$, f'_s is then defined as $f'_s \equiv F_s \left(\frac{c}{Z_s e} p'_{\varphi s}, \mathcal{H}'^{(1)}_s, p'_{\varphi s}, m'_s \right)$. The remaining gyrophase average in the last equation can be performed in a straightforward way using Eqs.(62) and (65).

Eq.(70) clearly also holds in the time interval (66), and so it determines the slow time-evolution for all times $\tau_{Ls} \ll t \ll \tau_C$. For consistency, the non-stationary Maxwell equations must also be solved with the same accuracy. Eq.(70) must be supplemented by appropriate boundary conditions: for open magnetic surfaces with boundaries prescribed on a given magnetic surface $\psi = const.$, at $\vartheta = \vartheta_1$ and $\vartheta = \vartheta_2$, with $\vartheta_1 < \vartheta_2$ and ϑ_1, ϑ_2 representing the internal and external boundaries, these are defined respectively either by prescribing $f_s(\vartheta_1) = f_s^{(1)}$ or $f_s(\vartheta_2) = f_s^{(2)}$ (see Fig.2 for a schematic view of the configuration geometry and the meaning of the notation). Both $f_s^{(1)}$ and $f_s^{(2)}$ are necessarily of the form (68) but their moments remain arbitrary in principle. As indicated below, this is essential for making comparisons with experimental observations.

The results obtained here have important consequences for the kinetic description of slow time-evolution of collisionless AD plasmas. Fluid fields and moment equations can be explicitly determined in terms of Eq.(70) by invoking the perturbative expansion outlined in Section 5 and the relations given in Section 6.

VIII. THE AMPERE EQUATION AND THE KINETIC DYNAMO

In this section we apply the kinetic solution for the QSA-KDF to discuss the properties of the Ampere equation and the implications for the self-generation of magnetic field by the quasi-stationary AD collisionless plasma. We refer here to this phenomenon as a *quasi-stationary kinetic dynamo effect*. Generalizing the treatment presented in Paper I, the Ampere equation for the self magnetic field becomes:

$$\nabla \times \mathbf{B}^{self} = \frac{4\pi}{c} \left(\mathbf{J}^T + \mathbf{J}^B + \mathbf{J}^P \right), \tag{72}$$

where $\mathbf{B}^{self}$ has been defined in Eq.(3) and here we have distinguished between the contributions arising from PPs, BPs and TPs, denoting the corresponding total current densities as $\mathbf{J}^T, \mathbf{J}^B$ and $\mathbf{J}^P$. As described in Sections 5 and 6, these fluid fields can be calculated in closed analytic form to the required order, by using the asymptotic expansion of the QSA-KDF. This gives:

$$\mathbf{J}^l \equiv \sum_{s=i,e} \mathbf{J}^l_s = \sum_{s=i,e} Z_s e \int_{\Gamma^l_u} d^3 v \mathbf{v} \widehat{f_{*s}} \cong$$

$$\cong \sum_{s=i,e} Z_s e \int_{\Gamma^l_u} d^3 v \mathbf{v} \left\{ \widehat{f}_s \left[1 + h^1_{Ds} + h^2_{Ds} \right] \right\} \tag{73}$$

for $l = T, B, P$ and where Γ^l_u denotes the appropriate velocity space domain of integration for trapped, bouncing and passing particles respectively. For convenience of notation, in the following we shall denote as $\mathbf{J} \equiv \mathbf{J}^T + \mathbf{J}^B + \mathbf{J}^P$ the total current density entering Eq.(72). It is possible to prove that in the case of open magnetic surfaces the total current density $\mathbf{J}$ in general has non-vanishing components along all of the three directions identified by the set of magnetic coordinates $(\psi, \varphi, \vartheta)$. Hence, $\mathbf{J}$ can be represented as

$$\mathbf{J} = \left(J_\psi \nabla\vartheta \times \nabla\varphi, J_\varphi \nabla\varphi, J_\vartheta \nabla\psi \times \nabla\varphi \right). \tag{74}$$

Let us now proceed with the study of the Ampere equation. The toroidal component of Eq.(72) gives, as usual, the generalized Grad-Shafranov equation for the poloidal flux function ψ_p:

$$\Delta^* \psi_p = -\frac{4\pi}{c} J_\varphi, \tag{75}$$

where the elliptic operator Δ^* is defined as $\Delta^* \equiv R^2 \nabla \cdot \left(R^{-2} \nabla \right)$. The remaining terms of Eq.(72) along the directions $\nabla\vartheta \times \nabla\varphi$ and $\nabla\psi \times \nabla\varphi$ give two equations for the toroidal component of the magnetic field I/R. These are respectively

$$\frac{\partial I}{\partial \psi} = \frac{4\pi}{c} J_\vartheta, \tag{76}$$

$$\frac{\partial I}{\partial \vartheta} = \frac{4\pi}{c} J_\psi, \tag{77}$$

yielding the constraint

$$\frac{\partial J_\psi}{\partial \psi} = \frac{\partial J_\vartheta}{\partial \vartheta} \tag{78}$$

which is a solubility condition for the structure functions. In this regard we notice that as a consequence of the kinetic constraints the function I in the previous equations is of the form $I(\psi, \vartheta, \varepsilon^k_M t)$, i.e., in contrast to Paper I it is no longer a flux-function. Therefore, the solubility condition (78) can always be satisfied. Eqs.(75)-(78) therefore provide consistent solutions for both poloidal and toroidal self magnetic fields in a collisionless AD plasma.

It is remarkable that in principle all of the populations of charged particles (PPs, BPs and TPs) can contribute to the generation of the toroidal magnetic field. More precisely, the following mechanisms can be involved:

#1) FLR and diamagnetic effects, driven by temperature anisotropy, of the same kind as those described in Paper I;

#2) Parallel velocity perturbations $U'_{\parallel *s}$, which generate a poloidal flow velocity, giving a related contribution to the electric current density through J_ψ and J_ϑ;

#3) FLR effects driven by the remaining thermodynamic forces (see Section 5). These contributions are produced by the diamagnetic KDF and arise because of the asymptotic ordering introduced here;

#4) Gyrophase-dependent contributions driven by the same thermodynamic forces. These are originated by the inverse GK transformation of the guiding-center quantities in the QSA-KDF.

As discussed above, contributions #2 and #4 were negligible under the circumstances discussed in Paper I. Therefore they should be considered as characteristic features of open-field configurations.

We refer to the mechanism of self-generation of both poloidal and toroidal magnetic fields as a *quasi-stationary kinetic dynamo effect*. In contrast to customary MHD treatments, this type of dynamo effect occurs *in the absence of possible instabilities or turbulence phenomena*. In particular, in the case of TPs, the self generation of toroidal field could take place even *without any net accretion* in the domain of interest, in presence of open magnetic field lines. This phenomenon is analogous to that treated in Paper I for closed-field configurations. In particular, the toroidal field is associated with the existence of torques which cause redistribution of angular momentum, producing radial inflows and outflows of disk material. As a consequence, various scenarios can be envisaged in which stationary radial flows and kinetic dynamos are present in AD plasmas, both affected by processes of type #1-#4.

IX. QUASI-STATIONARY ACCRETION FLOW

Let us now consider specifically the application of the kinetic solution developed here to the investigation of the accretion process in AD plasmas.

The inward accretion flow in ADs is usually "slow" in comparison with the characteristic Larmor time τ_{Ls}. For example, AD plasmas with $B \sim 10^1 - 10^8 G$ have Hydrogen-ion Larmor rotation times in the range $\tau_{Li} \sim 10^{-4} - 10^{-11}s$ which is shorter than the dynamical timescale at most relevant radii. For typical plasma densities and temperatures in the range $n_i \sim 10^9 - 10^{11}cm^{-3}$ and $T_i \sim 1 - 10keV$, the (Spitzer) ion collision time (below which the plasma can be considered collisionless) is in the range $\tau_C \sim 10^2 - 10^5s$ (the upper value corresponding to high temperature and low density). Independent of the physical origin of the accretion process, we can therefore expect that the present theory correctly describes phenomena occurring on all time-scales in the range $\tau_{Li} < t < \tau_C$.

We next determine the local poloidal and radial flow velocities for the various particle sub-species. By definition, these are given by

$$V_{ps} \equiv \mathbf{V}_s \cdot \mathbf{e}_p =$$
$$= \sum_{sub-species} \frac{1}{n_s^{tot}} \int_{\Gamma_u^l} d^3v \left[\mathbf{v} \cdot \mathbf{e}_p \right] \widehat{f_{*s}} \left[1 + g_s' \right], \tag{79}$$

$$V_{Rs} \equiv \mathbf{V}_s \cdot \mathbf{e}_R = \sum_{sub-species} \frac{1}{n_s^{tot}} J_{Rs}^l, \tag{80}$$

$$J_{Rs}^l \equiv \int_{\Gamma_u^l} d^3v \left[\mathbf{v} \cdot \mathbf{e}_R \right] \widehat{f_{*s}} \left[1 + g_s' \right], \tag{81}$$

where $\mathbf{e}_p \equiv \frac{\nabla\psi \times \nabla\varphi}{|\nabla\psi \times \nabla\varphi|}$ and $\mathbf{e}_R \equiv \frac{\nabla R}{|\nabla R|}$ and the summations are performed over the particle sub-species for $l = T, B, P$. We stress that the velocity-space integrals indicated above must contain the contributions from PPs, BPs and TPs and so $J_{Rs} = J_{Rs}^T + J_{Rs}^B + J_{Rs}^P$, where J_{Rs}^T, J_{Rs}^B and J_{Rs}^P are the corresponding mass currents. As an example, let us consider the leading-order contributions obtained ignoring FLR corrections. Explicit calculation gives

$$V_{ps} \cong U_{\|s} \mathbf{b} \cdot \mathbf{e}_p, \tag{82}$$
$$V_{Rs} \cong U_{\|s} \mathbf{b} \cdot \mathbf{e}_R, \tag{83}$$

where $U_{\|s} \equiv U_{\|s}\left(\psi, \vartheta, \varepsilon_M^k t\right) \equiv \frac{I}{B}\xi_s$ and the functional dependence $\xi_s = \xi_s\left(\psi, \vartheta, \varepsilon_M^k t\right)$ is prescribed by the kinetic constraints (38). We stress that the precise form of $\xi_s\left(\psi, \vartheta, \varepsilon_M^k t\right)$ still has to be chosen to satisfy the solubility constraints imposed by Ampere's law (see the discussion in previous section) and so the radial mass current density J_{Rs} is generally a function of the form $J_{Rs} \equiv J_{Rs}\left(\mathbf{x}, \varepsilon_M^k t\right) = \overline{J_{Rs}}(\psi, \vartheta, \varepsilon_M^k t)$. We are interested in situations where there is a *net radial accretion flow* i.e. where the average radial mass current $\langle\langle J_{Rs}\rangle\rangle \equiv \frac{1}{z_2-z_1}\int_{z_1}^{z_2} J_{Rs}dz$ (with z_1 and z_2 being suitably prescribed) is negative. There are local contributions to $\langle\langle J_{Rs}\rangle\rangle$ from TPs, BPs and PPs, but the overall accretion flow is mainly associated with PPs.

Let us show that such a solution exists. We seek particular Vlasov-Maxwell equilibria which are globally quasi-neutral, in the sense of Eq.(59). These equilibria are uniquely defined once $\Phi(\mathbf{x}, \varepsilon_M^k t)$, $\psi\left(\mathbf{x}, \varepsilon_M^k t\right)$, $I(\psi, \vartheta, \varepsilon_M^k t)$ and the structure functions are prescribed. The latter, by definition, are arbitrary smooth real functions of the specified variables as required by the kinetic constraints. Notice that, if the quasi-neutrality condition is valid, an analytical

solution for the ES potential can be obtained as shown in Paper I. Furthermore, we consider an example in which by assumption

1. $U_{\|s}$ is non-vanishing and dominant with respect to FLR effects, so that Eqs.(82) and (83) apply.

2. Particular solutions have a definite parity property with respect to the spatial reflection $z \to -z$. As a specific case, the poloidal flux ψ is assumed here to be antisymmetric, i.e., $\psi(z) = -\psi(-z)$, while both the toroidal and poloidal magnetic fields are symmetric. As a consequence, the toroidal current density must be antisymmetric. This can be realized only if a vertical electric field is present (i.e., one in the z direction), consistent with the quasi-neutrality condition.

If these assumptions are valid, Ampere's law demands that, to leading order, i.e., neglecting diamagnetic FLR effects,

$$\frac{\partial I}{\partial \psi} \cong \frac{4\pi}{c} \sum_s Z_s e n_s^{tot} U_{\|s}, \tag{84}$$

$$\frac{\partial I}{\partial \vartheta} \cong 0, \tag{85}$$

namely $I \cong I(\psi)$ at this order of approximation. Therefore, in this case a solution consistent with the requirement of net radial accretion flow and Vlasov-Maxwell equilibrium is obtained imposing that the species number density n_s^{tot} is even in ψ, while the species structure function ξ_s is odd with respect to the same variable. A solution of this type is consistent with the angular momentum conservation law (53); in order to obtain the solution, suitable kinetic boundary conditions must be prescribed (see the discussion following Eq.(71)). This proves that stationary accretion solutions exist and are admitted by the present kinetic theory for the "incoming" QSA-KDF, namely for $\widehat{f_{*s}}\Big|_{\vartheta_2}$ in the subset $\mathbf{v} \cdot \mathbf{e}_R < 0$ (see Fig.2). The same conclusion is in principle applicable for outflows, by appropriate prescription of the "outgoing" QSA-KDF $\widehat{f_{*s}}\Big|_{\vartheta_2}$ in the subset $\mathbf{v} \cdot \mathbf{e}_R > 0$. In fact, the angular momentum conservation law (53) allows both inward and outward radial fluid velocities for each species, namely having $\mathbf{V}_s^{tot} \cdot \mathbf{e}_R < 0$ or > 0 respectively. Indeed, for a collisionless plasma the species tensor pressure is generally non isotropic (see the related discussions in Paper I and Ref.[16]) such that Eq.(53) is identically satisfied. Unlike the customary view based on ideal MHD, for which a self-consistent treatment of inflow and outflow solutions is usually difficult, within the present theory both inflows and outflows can occur independently and are described consistently by their respective QSA-KDFs. In particular, Eq.(53) shows that radial flows arise due both to the parallel velocities $U_{\|s}$ and to the kinetic effects carried by the FLR diamagnetic corrections. As a result, species radial flow velocities appear necessarily in combination with non-isotropic tensor pressures and a non-vanishing toroidal magnetic field. In conclusion, the theory predicts the possibility of having purely inflowing matter in quasi-stationary AD plasmas, or of having co-existing inflows and outflows.

X.   CONCLUSIONS

In this paper, a consistent theoretical investigation of the slow kinetic dynamics of collisionless non-relativistic and axisymmetric AD plasmas has been presented. The formulation is based on a kinetic approach developed within the framework of the Vlasov-Maxwell description. We have considered here plasmas immersed in quasi-stationary magnetic fields characterized by open nested magnetic surfaces. This can be appropriate for radiatively inefficient accretion flows onto black holes, some of which are believed to be associated with a plasma of collisionless ions and electrons having different temperatures, and there can be other related applications to the inner regions of accretion flows onto magnetized neutron stars and white dwarfs. The discussion presented here provides a background for future investigations of instabilities and turbulence occurring in these plasmas.

We have shown that a new type of asymptotic kinetic equilibria exists, which can be described by QSA-KDFs expressed in terms of generalized bi-Maxwellian distributions. These solutions permit the consistent treatment of a number of physical properties characteristic of collisionless plasmas. The existence of these equilibrium solutions has been shown to be warranted by imposing suitable kinetic constraints for the structure functions entering the definition of the QSA-KDFs. In terms of these solutions, the slow dynamics of collisionless AD plasmas has been described by means of a suitable reduced GK-Vlasov equation. In addition, the theory permits the consistent treatment of

gravitational EM particle trapping phenomena, allowing one to distinguish between different populations of charged particles.

We have shown that the kinetic approach is suitable for the description of quasi-stationary AD plasmas subject to accretion flows and kinetic dynamo effects responsible for the self-generation of both poloidal and toroidal magnetic fields. Four intrinsically-kinetic physical mechanisms have been included in the treatment of this, related to temperature anisotropy, parallel velocity perturbations and FLR-diamagnetic effects.

The novelty of the present approach, with respect to traditional fluid treatments, lies in the possibility of explicitly constructing asymptotic solutions for the fluid equations: the calculation of all of the relevant fluid fields involved (e.g. the plasma charge and mass current densities and the radial flow velocity) can be performed in a straightforward way using a species-dependent asymptotic expansion of the QSA-KDF.

We believe that this study makes a relevant contribution for the description of two-temperature collisionless AD plasmas and the improvement of our understanding of their physical properties. The kinetic treatment developed here can also provide a convenient starting point for making a kinetic stability analysis of these plasmas.

XI.   ACKNOWLEDGMENTS

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